

Lipschitz Shadowing for Flows

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Abstract

Let ϕ be the flow generated by a smooth vector field X on a smooth closed manifold. We show that the Lipschitz shadowing property of ϕ is equivalent to the structural stability of X and that the Lipschitz periodic shadowing property of ϕ is equivalent to the Ω -stability of X .

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1 Introduction

The theory of shadowing of approximate trajectories (pseudotrajectories) of dynamical systems is now a well developed part of the global theory of dynamical systems (see, for example, the monographs [1], [2]). This theory is closely related to the classical theory of structural stability (the basic definitions of structural stability and Ω -stability for flows can be found, for example, in the monograph [3]). It is well known that a diffeomorphism has the shadowing property in a neighborhood of a hyperbolic set [4], [5] and a structurally stable diffeomorphism has the shadowing property on the whole manifold [6], [7], [8]. Analyzing the proofs of the first shadowing results by Anosov [4] and Bowen [5], it is easy to see that, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz (and the same holds in the case of a structurally stable diffeomorphism, see [2]). At the same time, it is easy to give an example of a diffeomorphism that is not structurally stable but has the shadowing property (see [9], for example). Thus, structural stability is not equivalent to shadowing. However it was shown in [10] that structural stability of a diffeomorphism is equivalent to Lipschitz shadowing.

Turning to flows, it is well known that a flow has the shadowing property in a neighborhood of a hyperbolic set [1], [2] and a structurally stable flow has the shadowing property on the whole manifold [2], [11]. In fact, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz and the same holds in the case of a structurally stable flow, see [2]. At the same time, it is easy to give an example of a flow that is not structurally stable but has the shadowing property (to construct such an example, one can use almost the same idea as in [9]). Thus, as with diffeomorphisms, structural stability is not equivalent to shadowing. However it is our purpose in this article to show that structural stability of a flow is equivalent to Lipschitz shadowing. Let us note that the proof for the flow case is a nontrivial modification of the proof for the diffeomorphism case.

One of the previously used approaches to compare shadowing property and structural stability is passing to C^1 -interiors. Sakai [12] showed that the C^1 -interior of the set of diffeomorphisms with the shadowing property coincides with the set of structurally stable diffeomorphisms. See also [13] for the generalization of this result to other types of shadowing properties. For vector fields the situation is different. There is an example of a vector field with the robust shadowing property which is not structurally stable [14]. See also [15, 16, 17] for some positive results in this direction.

In this paper, we also study vector fields having the Lipschitz periodic shadowing property. Diffeomorphisms having the Lipschitz periodic shadowing property were studied in [18], where it was shown that this property is equivalent to Ω -stability. We prove a similar statement for vector fields.

2 Preliminaries

Let M be a smooth closed manifold with Riemannian metric $\text{dist}(\cdot, \cdot)$ and let X be a vector field on M of class C^1 . Let $\phi(t, x)$ be the flow on M generated by X .

Definition 1. A (not necessarily continuous) function $y : I \rightarrow M$ (where I is an interval in \mathbb{R}) is called a d -pseudotrajectory if

$$\text{dist}(y(\tau + t), \phi(t, y(\tau))) \leq d, \quad 0 \leq t \leq 1, \quad \tau, \tau + t \in I.$$

Mostly we work with pseudotrajectories defined on $I = \mathbb{R}$.

Definition 2. We say that the vector field X has the *Lipschitz shadowing property* ($X \in \text{LipSh}$) if there exist d_0 and $\mathcal{L} > 0$ such that if $y : \mathbb{R} \mapsto M$ is a d -pseudotrajectory for $d \leq d_0$, then $y(t)$ is $\mathcal{L}d$ -shadowed by a trajectory, that is, there exists a trajectory $x(t)$ of X and an increasing homeomorphism (reparametrization) $\alpha(t)$ of the real line satisfying

$$\alpha(0) = 0, \quad \left| \frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1} - 1 \right| \leq \mathcal{L}d \quad (1)$$

for $t_2 \neq t_1$ and

$$\text{dist}(y(t), x(\alpha(t))) \leq \mathcal{L}d \quad (2)$$

for all t .

Definition 3. We say that the vector field X has the *Lipschitz periodic shadowing property* ($X \in \text{LipPerSh}$) if there exist d_0 and $\mathcal{L} > 0$ such that if $y : \mathbb{R} \mapsto M$ is a periodic d -pseudotrajectory for $d \leq d_0$, then $y(t)$ is $\mathcal{L}d$ -shadowed by a periodic trajectory, that is, there exists a trajectory $x(t)$ of X and an increasing homeomorphism $\alpha(t)$ of the real line satisfying inequalities (1) and (2) and such that

$$x(t + \omega) = x(t)$$

for some $\omega > 0$.

The last equality implies that $x(t)$ is either a closed trajectory or a rest point of the flow ϕ .

The main results of the paper are the following theorems.

Theorem 1. *A vector field X satisfies the Lipschitz shadowing property if and only if X is structurally stable.*

Theorem 2. *A vector field X satisfies the Lipschitz periodic shadowing property if and only if X is Ω -stable.*

It is known that expansive diffeomorphisms having the Lipschitz shadowing property are Anosov (see [10]).

We show, as a consequence of Theorem 1, that expansive vector fields having the Lipschitz shadowing property are Anosov. Let us recall the definition of expansivity for vector fields.

Definition 4. We say that a vector field X and the corresponding flow $\phi(t, x)$ are *expansive* if there exist constants $a, \delta > 0$ such that if

$$\text{dist}(\phi(t, x), \phi(\alpha(t), y)) < a, \quad t \in \mathbb{R},$$

for points $x, y \in M$ and an increasing homeomorphism α of the real line, then $y = \phi(\tau, x)$ for some $|\tau| < \delta$.

Theorem 3. *An expansive vector field X having the Lipschitz shadowing property is Anosov.*

Proof. By Theorem 1, a vector field X having the Lipschitz shadowing property is structurally stable. Hence, there exists a neighborhood \mathcal{N} of X in the C^1 -topology such that any vector field in \mathcal{N} is expansive (this property of X is sometimes called robust expansivity).

By Theorem B of [19], robustly expansive vector fields having the shadowing property are Anosov. \square

In Sec. 3 we prove Theorem 1 and in Sec. 4 we prove Theorem 2. Both proofs are long so that each section is divided into several subsections.

3 The Lipschitz shadowing property

As was mentioned above in [11] it was proved that structurally stable vector fields have the Lipschitz shadowing property. Our goal here is to show that vector fields satisfying Lipschitz shadowing are structurally stable. It is well known (see [20]) that for this purpose it is enough to show that such a vector field satisfies Axiom A' and the strong transversality condition.

First we show that Lipschitz shadowing implies discrete Lipschitz shadowing. Define a diffeomorphism f on M by setting $f(x) = \phi(1, x)$.

Definition 5. The vector field X has the *discrete Lipschitz shadowing property* if there exist $d_0, L > 0$ such that if $y_k \in M$ is a sequence with

$$\text{dist}(y_{k+1}, f(y_k)) \leq d, \quad k \in \mathbb{Z}$$

for $d \leq d_0$, then there exist sequences $x_k \in M$ and $t_k \in \mathbb{R}$ satisfying

$$|t_k - 1| \leq Ld, \quad \text{dist}(x_k, y_k) \leq Ld, \quad x_{k+1} = \phi(t_k, x_k)$$

for all k .

Lemma 1. *Lipschitz shadowing implies discrete Lipschitz shadowing.*

Proof. Let y_k be a sequence with

$$\text{dist}(y_{k+1}, f(y_k)) = \text{dist}(y_{k+1}, \phi(1, y_k)) \leq d, \quad k \in \mathbb{Z}.$$

Then we define

$$y(t) = \phi(t - k, y_k) \quad k \leq t < k + 1, \quad k \in \mathbb{Z}.$$

Assume that $k \leq \tau < k + 1$. If $0 \leq t \leq 1$ and $\tau + t < k + 1$, then

$$\text{dist}(y(\tau + t), \phi(t, y(\tau))) = \text{dist}(\phi(\tau + t - k, y_k), \phi(t, \phi(\tau - k, y_k))) = 0$$

and if $k + 1 \leq \tau + t$, then

$$\begin{aligned} & \text{dist}(y(\tau + t), \phi(t, y(\tau))) \\ &= \text{dist}(\phi(\tau + t - k - 1, y_{k+1}), \phi(t + \tau - k, y_k)) \\ &= \text{dist}(\phi(\tau + t - k - 1, y_{k+1}), \phi(\tau + t - k - 1, \phi(1, y_k))) \\ &\leq \nu d, \end{aligned}$$

where ν is a constant such that

$$\text{dist}(\phi(t, x), \phi(t, y)) \leq \nu \text{dist}(x, y) \quad \text{for } x, y \in M, 0 \leq t \leq 1. \quad (3)$$

Then if $d \leq d_0/\nu$, there exists a trajectory $x(t)$ of X and a function $\alpha(t)$ satisfying

$$\left| \frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1} - 1 \right| \leq \mathcal{L}\nu d$$

for $t_2 \neq t_1$ and

$$\text{dist}(y(t), x(\alpha(t))) \leq \mathcal{L}\nu d$$

for all t . Then if we define

$$x_k = x(\alpha(k)), \quad t_k = \alpha(k+1) - \alpha(k),$$

we see that

$$x_{k+1} = x(\alpha(k+1)) = \phi(\alpha(k+1) - \alpha(k), x(\alpha(k))) = \phi(t_k, x_k),$$

$$\text{dist}(x_k, y_k) = \text{dist}(x(\alpha(k)), y(k)) \leq \mathcal{L}\nu d$$

and

$$|t_k - 1| = \left| \frac{\alpha(k+1) - \alpha(k)}{k+1 - k} - 1 \right| \leq \mathcal{L}\nu d.$$

Taking $L = \mathcal{L}\nu$ and d_0 in Definition 5 as d_0/ν , we complete the proof of the lemma. \square

Our main tool in the proof is the following lemma which relates the shadowing problem to the problem of existence of bounded solutions of certain difference equations. To “linearize” our problem, we apply the standard technique of exponential mappings.

Denote by $T_x M$ the tangent space to M at a point x ; let $|v|$ be the norm of v corresponding to the metric $\text{dist}(\cdot, \cdot)$.

Let $\exp : TM \mapsto M$ be the standard exponential mapping on the tangent bundle of M and let \exp_x be the corresponding mapping $T_x M \mapsto M$.

Denote by $B(r, x)$ the ball in M of radius r centered at a point x and by $B_T(r, x)$ the ball in $T_x M$ of radius r centered at the origin.

There exists $r > 0$ such that, for any $x \in M$, \exp_x is a diffeomorphism of $B_T(r, x)$ onto its image, and \exp_x^{-1} is a diffeomorphism of $B(r, x)$ onto its image. In addition, we may assume that r has the following property:

If $v, w \in B_T(r, x)$, then

$$\text{dist}(\exp_x(v), \exp_x(w)) \leq 2|v - w|; \quad (4)$$

if $y, z \in B(r, x)$, then

$$|\exp_x^{-1}(y) - \exp_x^{-1}(z)| \leq 2 \text{dist}(y, z). \quad (5)$$

Let $x(t)$ be a trajectory of X ; set $p_k = x(k)$ for $k \in \mathbb{Z}$. Denote $A_k = Df(p_k)$ and $\mathcal{M}_k = T_{p_k}M$. Clearly, A_k is a linear isomorphism between \mathcal{M}_k and \mathcal{M}_{k+1} .

In the sequel whenever we construct d -pseudotrajectories of the diffeomorphism f , we always take d so small that the points of the pseudotrajectories under consideration, the points of the associated shadowing trajectories, their lifts to tangent spaces, etc. belong to the corresponding balls $B(r, p_k)$ and $B_T(r, p_k)$.

We consider the mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : B_T(\rho, p_k) \rightarrow \mathcal{M}_{k+1} \quad (6)$$

with $\rho \in (0, r)$ small enough, so that

$$f \circ \exp_{p_k}(B_T(\rho, p_k)) \subset B(r, p_{k+1}).$$

It follows from standard properties of the exponential mapping that $D \exp_x(0) = \text{Id}$; hence,

$$DF_k(0) = A_k.$$

Since M is compact, for any $\mu > 0$ we can find $\delta = \delta(\mu) > 0$ such that if $|v| \leq \delta$, then

$$|F_k(v) - A_k v| \leq \mu |v|. \quad (7)$$

Lemma 2. *Assume that X has the discrete Lipschitz shadowing property with constant L . Let $x(t)$ be an arbitrary trajectory of X , let $p_k = x(k)$, $A_k = Df(p_k)$ and let $b_k \in \mathcal{M}_k$ be a bounded sequence (denote $b = \|b\|_\infty$). Then there exists a sequence s_k of scalars with $|s_k| \leq b' = L(2b + 1)$ such that the difference equation*

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k + b_{k+1}$$

has a solution v_k such that

$$\|v\|_\infty \leq 2b'.$$

Proof. Fix a natural number N and define $\Delta_k \in \mathcal{M}_k$ as the solution of

$$v_{k+1} = A_k v_k + b_{k+1}, \quad k = -N, \dots, N-1$$

with $\Delta_{-N} = 0$. Then

$$|\Delta_k| \leq C, \quad (8)$$

where C depends on N , b and an upper bound on $|A_k|$.

Fix a small number $d > 0$ and fix μ in (7) so that $\mu < 1/(2C)$. Then consider the sequence of points $y_k \in M$, $k \in \mathbb{Z}$, defined as follows: $y_k = p_k$ for $k \leq -N$, $y_k = \exp_{p_k}(d\Delta_k)$ for $-N+1 \leq k \leq N$, and $y_{N+k} = f^k(y_N)$ for $k > 0$.

By definition, $y_{k+1} = f(y_k)$ for $k \leq -N$ and $k \geq N$. If $-N-1 \leq k \leq N-1$, then

$$y_{k+1} = \exp_{p_{k+1}}(d\Delta_{k+1}) = \exp_{p_{k+1}}(dA_k\Delta_k + db_{k+1}),$$

and it follows from estimate (4) that if d is small enough, then

$$\text{dist}\left(y_{k+1}, \exp_{p_{k+1}}(dA_k\Delta_k)\right) \leq 2d|b_{k+1}| \leq 2db. \quad (9)$$

On the other hand,

$$f(y_k) = \exp_{p_{k+1}}(F_k(d\Delta_k))$$

(see the definition (6) of the mapping F_k), and we deduce from (4), (7) and (8) that if $Cd \leq \delta(\mu)$

$$\begin{aligned} \text{dist}\left(f(y_k), \exp_{p_{k+1}}(dA_k\Delta_k)\right) &\leq 2|F_k(d\Delta_k) - dA_k\Delta_k| \\ &\leq 2\mu|d\Delta_k| \\ &\leq 2C\mu d \\ &< d. \end{aligned} \quad (10)$$

Estimates (9) and (10) imply that

$$\text{dist}(y_{k+1}, f(y_k)) < d(2b+1), \quad k \in \mathbb{Z},$$

if d is small enough (let us emphasize here that the required smallness of d depends on b , N and estimates on A_k). By hypothesis, there exist sequences x_k and t_k such that

$$|t_k - 1| \leq b'd, \quad \text{dist}(x_k, y_k) \leq b'd, \quad x_{k+1} = \phi(t_k, x_k), \quad k \in \mathbb{Z}.$$

If we write

$$x_k = \exp_{p_k}(dc_k), \quad t_k = 1 + ds_k,$$

then it follows from estimate (5) that

$$|dc_k - d\Delta_k| \leq 2 \operatorname{dist}(x_k, y_k) \leq 2b'd.$$

Thus,

$$|c_k - \Delta_k| \leq 2b', \quad k \in \mathbb{Z}. \quad (11)$$

Clearly,

$$|s_k| \leq b', \quad k \in \mathbb{Z}. \quad (12)$$

We may assume that the value ρ fixed above is small enough, so that the mappings

$$G_k : (-\rho, \rho) \times B_T(\rho, p_k) \rightarrow \mathcal{M}_{k+1}$$

given by

$$G_k(t, v) = \exp_{p_{k+1}}^{-1}(\phi(1 + t, \exp_{p_k}(v))).$$

are defined. Then $G_k(0, 0) = 0$,

$$D_t G_k(t, v)|_{t=0, v=0} = X(p_{k+1}), \quad D_v G_k(t, v)|_{t=0, v=0} = A_k. \quad (13)$$

We can write the equality

$$x_{k+1} = \phi(1 + ds_k, x_k)$$

in the form

$$\exp_{p_{k+1}}(dc_{k+1}) = \phi(1 + ds_k, \exp_{p_k}(dc_k)),$$

which is equivalent to

$$dc_{k+1} = G_k(ds_k, dc_k). \quad (14)$$

Now let $d = d_m$, where $d_m \rightarrow 0$. Note that the corresponding $c_k = c_k^{(m)}$, $t_k = t_k^{(m)}$, and $s_k = s_k^{(m)}$ depend on m .

Since $|c_k^{(m)}| \leq 2b' + C$ and $|s_k^{(m)}| \leq b'$ for all $m \geq 1$ and $-N \leq k \leq N-1$, by taking a subsequence if necessary, we can assume that $c_k^{(m)} \rightarrow \tilde{c}_k$, $t_k^{(m)} \rightarrow \tilde{t}_k$, and $s_k^{(m)} \rightarrow \tilde{s}_k$ for $-N \leq k \leq N-1$ as $m \rightarrow \infty$.

Applying relations (14) and (13), we can write

$$d_m c_{k+1}^{(m)} = G_k(d_m s_k^{(m)}, d_m c_k) = A_k d_m c_k^{(m)} + X(p_{k+1}) d_m s_k^{(m)} + o(d_m).$$

Dividing by d_m , we get the relations

$$c_{k+1}^{(m)} = A_k c_k^{(m)} + X(p_{k+1}) s_k^{(m)} + o(1), \quad -N \leq k \leq N-1.$$

Letting $m \rightarrow \infty$, we arrive at

$$\tilde{c}_{k+1} = A_k \tilde{c}_k + X(p_{k+1}) \tilde{s}_k, \quad -N \leq k \leq N-1,$$

where

$$|\Delta_k - \tilde{c}_k| \leq 2b', \quad |\tilde{s}_k| \leq b', \quad -N \leq k \leq N-1$$

due to (11) and (12).

Denote the obtained \tilde{s}_k by $s_k^{(N)}$. Then $v_k^{(N)} = \Delta_k - \tilde{c}_k$ is a solution of the system

$$v_{k+1}^{(N)} = A_k v_k^{(N)} + X(p_{k+1}) s_k^{(N)} + b_{k+1}, \quad -N \leq k \leq N-1,$$

such that $|v_k^{(N)}| \leq 2b'$.

There exist subsequences $s_k^{(j_N)} \rightarrow s'_k$ and $v_k^{(j_N)} \rightarrow v'_k$ as $N \rightarrow \infty$ (we do not assume uniform convergence) such that $|s'_k| \leq b'$, $|v'_k| \leq 2b'$, and

$$v'_{k+1} = A_k v'_k + X(p_{k+1}) s'_k + b_{k+1}, \quad k \in \mathbb{Z}.$$

Thus, the lemma is proved. \square

Further, we have to refer to two known statements. It is convenient to state them as lemmas. First we make a definition.

Definition 6. Consider a sequence of linear isomorphisms

$$C = \{C_k : \mathbb{R}^n \mapsto \mathbb{R}^n, k \in \mathbb{Z}\}$$

such that $\sup_k (\|C_k\|, \|C_k^{-1}\|) < \infty$. The associated *transition operator* is defined for indices $k, l \in \mathbb{Z}$ by

$$\Phi(k, l) = \begin{cases} C_{k-1} \circ \dots \circ C_l, & l < k, \\ \text{Id}, & l = k, \\ C_k^{-1} \circ \dots \circ C_{l-1}^{-1}, & l > k. \end{cases}$$

The sequence C is called *hyperbolic on \mathbb{Z}_+* (has an *exponential dichotomy on \mathbb{Z}_+*) if there exist constants $K > 0$, $\lambda \in (0, 1)$, and families of linear subspaces S_k, U_k of \mathbb{R}^n for $k \in \mathbb{Z}_+$ such that

- (1) $S_k \oplus U_k = \mathbb{R}^n$, $k \in \mathbb{Z}_+$;
- (2) $C_k(S_k) = S_{k+1}$ and $C_k(U_k) = U_{k+1}$ for $k \in \mathbb{Z}_+$;
- (3) $|\Phi(k, l)v| \leq K\lambda^{k-l}|v|$ for $v \in S_l$, $k \geq l \geq 0$;
- (4) $|\Phi(k, l)v| \leq K\lambda^{l-k}|v|$ for $v \in U_l$, $0 \leq k \leq l$.

The following result was shown by Maizel' [21] (see also Coppel [22]).

Lemma 3. *If the system*

$$v_{k+1} = C_k v_k + b_{k+1}, \quad k \geq 0,$$

has a bounded solution v_k for any bounded sequence b_k , then the sequence C is hyperbolic on \mathbb{Z}_+ (and a similar statement holds for \mathbb{Z}_-).

The second of the results which we need was proved by Pliss [23]. An analogous statement was proved later by Palmer [24, 25]; he also described the Fredholm properties of the corresponding operator

$$\{v_k \in \mathbb{R}^m : k \in \mathbb{Z}\} \mapsto \{v_k - A_{k-1}v_{k-1}\}.$$

Lemma 4. *Set*

$$B^+(C) = \{v \in \mathbb{R}^n : |\Phi(k, 0)v| \rightarrow 0, k \rightarrow +\infty\}$$

and

$$B^-(C) = \{v \in \mathbb{R}^n : |\Phi(k, 0)v| \rightarrow 0, k \rightarrow -\infty\}.$$

Then the following two statements are equivalent:

- (a) *for any bounded sequence $\{b_k \in \mathbb{R}^n, k \in \mathbb{Z}\}$ there exists a bounded sequence $\{v_k \in \mathbb{R}^n, k \in \mathbb{Z}\}$ such that*

$$v_{k+1} = C_k v_k + b_{k+1}, \quad k \in \mathbb{Z};$$

- (b) *the sequence C is hyperbolic on each of the rays \mathbb{Z}_+ and \mathbb{Z}_- , and the subspaces $B^+(C)$ and $B^-(C)$ are transverse.*

Remark 1. Both Lemmas 3 and 4 were proved for linear systems of differential equations, but they hold as well (in the form stated above) for sequences of linear isomorphisms of Euclidean spaces and for sequences of linear isomorphisms of arbitrary linear spaces of the same dimension (we apply them to the isomorphisms A_k of the spaces \mathcal{M}_k in Section 3.2 and to the isomorphisms B_k of the spaces V_k in Sections 3.3 and 3.4). For further discussion of this point, see [26].

In the following three sections we assume that X has the Lipschitz shadowing property (and, consequently, the discrete Lipschitz shadowing property).

3.1 Hyperbolicity of the rest points

Let x_0 be a rest point. We apply Lemma 2 with $p_k = x_0$. Noting that $X(p_k) = 0$, we conclude that the difference equation

$$v_{k+1} = Df(x_0)v_k + b_{k+1}$$

has a bounded solution v_k for all bounded sequences $b_k \in \mathcal{M}_{x_0}$. Then it follows from Lemma 4 that

$$v_{k+1} = Df(x_0)v_k$$

is hyperbolic on both \mathbb{Z}_+ and \mathbb{Z}_- . In particular, this implies that any solution bounded on \mathbb{Z}_+ tends to 0 as $k \rightarrow \infty$. However if $Df(x_0)$ had an eigenvalue on the unit circle, the equation would have a nonzero solution with constant norm. Hence the eigenvalues of $Df(x_0)$ lie off the unit circle. So x_0 is hyperbolic.

3.2 The rest points are isolated in the chain recurrent set

Lemma 5. *If a rest point x_0 is not isolated in the chain recurrent set \mathcal{CR} , then there is a homoclinic orbit $x(t)$ associated with it.*

Proof. We choose $d > 0$ so small that $\text{dist}(\phi(t, y), x_0) \leq \mathcal{L}d$ for $|t|$ large implies that $\phi(t, y) \rightarrow x_0$ as $|t| \rightarrow \infty$.

Assume that there exists a point $y \in \mathcal{CR}$ such that $y \neq x_0$ is arbitrarily close to x_0 . Since y is chain recurrent, given any ε_0 and $\theta > 0$ we can find points y_1, \dots, y_N and numbers $T_0, \dots, T_N > \theta$ such that

$$\text{dist}(\phi(T_0, y), y_1) < \varepsilon_0,$$

$$\text{dist}(\phi(T_i, y_i), y_{i+1}) < \varepsilon_0, \quad i = 1, \dots, N,$$

$$\text{dist}(\phi(T_N, y_N), y) < \varepsilon_0.$$

Set $T = T_0 + \dots + T_N$ and define g^* on $[0, T]$ by

$$g^*(t) = \begin{cases} \phi(t, y), & 0 \leq t < T_0, \\ \phi(t, y_i), & T_0 + \dots + T_{i-1} \leq t < T_0 + \dots + T_i, \\ y, & t = T. \end{cases}$$

Clearly, for any $\varepsilon > 0$ we can find ε_0 depending only on ε and ν (see (3)) such that $g^*(t)$ is an ε -pseudotrajectory on $[0, T]$.

Then we define

$$g(t) = \begin{cases} x_0, & t \leq 0, \\ g^*(t), & 0 < t \leq T, \\ x_0, & t > T. \end{cases}$$

We want to choose y and ε in such a way that $g(t)$ is a d -pseudotrajectory. We need to show that for all τ and $0 \leq t \leq 1$

$$\text{dist}(\phi(t, g(\tau)), g(\tau + t)) \leq d. \tag{15}$$

Clearly this holds for (i) $\tau \leq -1$, (ii) $\tau \geq T$, (iii) $\tau, \tau + t \in [-1, 0]$, and (iv) $\tau, \tau + t \in [0, T]$.

If $-1 \leq \tau \leq 0$, $\tau + t > 0$, then with ν as in (3)

$$\begin{aligned} \text{dist}(\phi(t, g(\tau)), g(\tau + t)) &= \text{dist}(x_0, g^*(\tau + t)) \\ &\leq \text{dist}(x_0, \phi(\tau + t, y)) + \text{dist}(\phi(\tau + t, y), g^*(\tau + t)) \\ &\leq \nu \text{dist}(x_0, y) + \varepsilon \\ &\leq d, \end{aligned}$$

if $\text{dist}(y, x_0)$ and ε are sufficiently small. Note that, for the fixed y , we can decrease ε and increase N, T_0, \dots, T_N arbitrarily so that $g(t)$ remains a d -pseudotrajectory.

Similarly, (15) holds if $\tau \in [0, T]$ and $\tau + t > T$.

Thus $g(t)$ is $\mathcal{L}d$ -shadowed by a trajectory $x(t)$ so that in particular $\text{dist}(x(t), x_0) \leq \mathcal{L}d$ if $|t|$ is sufficiently large so that $x(t) \rightarrow x_0$ as $|t| \rightarrow \infty$.

We must also be sure that $x(t) \neq x_0$. If y is not on the local stable manifold of x_0 , then there exists $\varepsilon_1 > 0$ independent of y such that $\text{dist}(\phi(t_0, y), x_0) \geq \varepsilon_1$ for some $t_0 > 0$. We can choose $T_0 > t_0$. Now we know that $\text{dist}(x(t), \phi(t_0, y)) \leq \mathcal{L}d$. So provided $\mathcal{L}d < \varepsilon_1$, we have $x(t_0) \neq x_0$.

If y is on the local stable manifold of x_0 , then provided $\text{dist}(y, x_0)$ is sufficiently small, it is not on the local unstable manifold of x_0 . Then, applying the same argument to the flow with time reversed noting that the chain recurrent set is also the chain recurrent set for the reversed flow and also that the reversed flow will have the Lipschitz shadowing property also, we show that $x(t) \neq x_0$.

Now we show the existence of this homoclinic orbit $x(t)$ leads to a contradiction. Set $p_k = x(k)$. Since $A_k X(p_k) = X(p_{k+1})$, it is easily verified that if

$$\beta_{k+1} = \beta_k + s_k, \quad k \in \mathbb{Z}$$

then $v_k = \beta_k X(p_k)$ is a solution of

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k, \quad k \in \mathbb{Z}. \quad (16)$$

Also if s_k is bounded then $\beta_k X(p_k)$ is also bounded, since $X(p_k) \rightarrow 0$ exponentially as $|k| \rightarrow \infty$ and $|\beta_k|/|k|$ is bounded.

By Lemma 2, for all bounded $b_k \in \mathcal{M}_k$ there exists a bounded scalar sequence s_k such that

$$v_{k+1} = A_k v_k + X(p_{k+1}) s_k + b_{k+1}$$

has a bounded solution. But we know (16) has a bounded solution. It follows that

$$v_{k+1} = A_k v_k + b_{k+1}$$

has a bounded solution for arbitrary $b_k \in \mathcal{M}_k$. Then it follows from Lemma 4 that

$$v_{k+1} = A_k v_k$$

is hyperbolic on both \mathbb{Z}_+ and \mathbb{Z}_- and that the spaces $B^+(A)$ and $B^-(A)$ are transverse. This is a contradiction since $\dim B^+(A) + \dim B^-(A) = n$ (because $B^+(A)$ has the same dimension as the stable manifold of x_0 and

$B^-(A)$ has the same dimension as the unstable manifold of x_0) but they contain $X(p_0) \neq 0$ in their intersection.

So we conclude that the rest points are isolated in the chain recurrent set. \square

3.3 Hyperbolicity of the chain recurrent set

We have shown that the rest points of X are hyperbolic and form a finite, isolated subset of the chain recurrent set \mathcal{CR} . Let Σ be the chain recurrent set minus the rest points. We want to show this set is hyperbolic. To this end we use the following lemma. Let us first introduce some notation.

Let $x(t)$ be a trajectory of X in Σ . Put $p_k = x(k)$ and denote by P_k the orthogonal projection in \mathcal{M}_k with kernel spanned by $X(p_k)$ and by V_k the orthogonal complement to $X(p_k)$ in \mathcal{M}_k . Introduce the operators $B_k = P_{k+1}A_k : V_k \mapsto V_{k+1}$.

Lemma 6. *For every bounded sequence $b_k \in V_k$ (denote $b = \|b\|_\infty$) there exists a solution $v_k \in V_k$ of the system*

$$v_{k+1} = B_k v_k + b_{k+1}, \quad k \in \mathbb{Z}, \quad (17)$$

such that for all k ,

$$|v_k| \leq 2L(2b + 1).$$

Proof. By Lemma 2, there exists a bounded sequence s_k such that the system

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k + b_{k+1}, \quad k \in \mathbb{Z} \quad (18)$$

has a solution w_k with $|w_k| \leq 2L(2b + 1)$.

Note that $A_k X(p_k) = X(p_{k+1})$. Since $(\text{Id} - P_k)v \in \{X(p_k)\}$ for $v \in \mathcal{M}_k$, we see that $P_{k+1}A_k(\text{Id} - P_k) = 0$, which gives us the equality

$$P_{k+1}A_k = P_{k+1}A_k P_k. \quad (19)$$

Multiplying (18) by P_{k+1} , taking into account the equalities $P_{k+1}X(p_{k+1}) = 0$ and $P_{k+1}b_{k+1} = b_{k+1}$, and applying (19), we see that $v_k = P_k w_k$ is the required solution.

Thus, the lemma is proved. \square

Now we prove Σ is hyperbolic. Let $x(t)$ be a trajectory in Σ with the same notation as given before Lemma 6. Then by Lemmas 6 and 4,

$$v_{k+1} = B_k v_k, \quad v_k \in V_k \quad (20)$$

is hyperbolic on both \mathbb{Z}_+ and \mathbb{Z}_- and $B^+(B)$ and $B^-(B)$ are transverse. It follows that the adjoint system

$$v_{k+1} = (B_k)^{* -1} v_k, \quad v_k \in V_k$$

is hyperbolic on both \mathbb{Z}_+ and \mathbb{Z}_- and has no nontrivial bounded solution.

Now we consider the discrete linear skew product flow on the normal bundle V over Σ generated by the map defined for $p \in \Sigma$, $v \in V_p$ (where V_p is the orthogonal complement to $X(p)$ in $T_p M$) by

$$(p, v) \mapsto (\phi(1, p), B_p v), \quad (21)$$

where $B_p = P_{\phi(1, p)} D\phi(1, p)$, P_p being the orthogonal projection of $T_p M$ onto V_p . Its adjoint flow is generated by the map defined by

$$(p, v) \mapsto (\phi(1, p), (B_p^*)^{-1} v).$$

Now we want to apply the Corollary on page 492 in Sacker and Sell [27]. What we have shown above is that the adjoint flow has the no nontrivial bounded solution property. It follows from the Sacker and Sell corollary that the adjoint flow is hyperbolic and hence the original skew product flow

$$(p, v) \mapsto (\phi(1, p), B_p v)$$

is also. However then it follows from Theorem 3 in Sacker and Sell [28] that Σ is hyperbolic.

3.4 Strong Transversality

To verify strong transversality, let $x(t)$ be a trajectory that belongs to the intersection of the stable and unstable manifolds of two trajectories, $x_+(t)$ and $x_-(t)$, respectively, lying in the chain recurrent set. Denote $p_0 = x(0)$ and $p_k = x(k)$, $k \in \mathbb{Z}$; let $W^s(p_0)$ and $W^u(p_0)$ denote the stable manifold of $x_+(t)$ and the unstable manifold of $x_-(t)$, respectively. Denote by E^s and E^u the tangent spaces of $W^s(p_0)$ and $W^u(p_0)$ at p_0 .

By Lemma 6 (using the same notation as in the previous section), for all bounded $b_k \in V_k$, there exists a bounded solution $v_k \in V_k$ of (17). By Lemma 4 again, this implies that

$$\mathcal{E}^s + \mathcal{E}^u = V_0, \quad (22)$$

where

$$\mathcal{E}^s = \{w_0 : w_{k+1} = B_k w_k, \sup_{k \geq 0} |w_k| < \infty\},$$

$$\mathcal{E}^u = \{w_0 : w_{k+1} = B_k w_k, \sup_{k \leq 0} |w_k| < \infty\}$$

Moreover (20) is hyperbolic on both \mathbb{Z}_+ and \mathbb{Z}_- .

We are going to use the following folklore result, which for completeness we prove after showing it implies the strong transversality:

$$\mathcal{E}^s \subset E^s, \quad \mathcal{E}^u \subset E^u. \quad (23)$$

Combining equality (22) with the inclusions (23) and the trivial relations

$$E^s = V_0 \cap E^s + \{X(p_0)\}, \quad E^u = V_0 \cap E^u + \{X(p_0)\},$$

we conclude that

$$E^s + E^u = T_{p_0} M,$$

and so the strong transversality holds.

Let us now prove the first relation in (23); the second one can be proved in a similar way.

Case 1: The limit trajectory in \mathcal{CR} is a rest point. In this case, the stable manifold of the rest point coincides with its stable manifold as a fixed point of the time-one map $f(x) = \phi(1, x)$. By the theory for diffeomorphisms, if p_k is a trajectory on the stable manifold, the tangent space to the stable manifold at p_0 is the subspace E^s of initial values of bounded solutions of

$$v_{k+1} = A_k v_k, \quad k \geq 0. \quad (24)$$

Let us prove that $\mathcal{E}^s \subset E^s$. Fix an arbitrary sequence w_k satisfying $w_{k+1} = B_k w_k$ with $w_0 \in \mathcal{E}^s$. Consider the sequence

$$v_k = \lambda_k X(p_k) / |X(p_k)| + w_k$$

with λ_k satisfying

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|} \lambda_k - \frac{X(p_{k+1})^*}{|X(p_{k+1})|} A_k w_k \quad (25)$$

and $\lambda_0 = 0$. It is easy to see that v_k satisfy (24).

Since $x(t)$ is on the stable manifold of a hyperbolic rest point, there are positive constants K and α such that

$$|\dot{x}(t)| \leq K e^{-\alpha(t-s)} |\dot{x}(s)|$$

for $0 \leq s \leq t$. From this it follows that

$$|X(p_k)| \leq K e^{-\alpha(k-m)} |X(p_m)|$$

for $0 \leq m \leq k$ so that the scalar difference equation

$$\lambda_{k+1} = \frac{|X(p_{k+1})|}{|X(p_k)|} \lambda_k$$

is hyperbolic on \mathbb{Z}_+ and is, in fact, stable. Since the second term on the right-hand side of equation (25) is bounded as $k \rightarrow \infty$, it follows that λ_k are bounded for any choice of λ_0 . This fact implies that v_k is a bounded solution of (24), and we conclude that $v_0 = w_0 \in E^s$, hence $\mathcal{E}^s \subset E^s$.

The proof in Case 1 is complete.

Case 2: Assume that the limit trajectory is in Σ , the chain recurrent set minus the fixed points which we know to be hyperbolic. We want to find the intersection of its stable manifold near $p_0 = x(0)$ with the cross-section at p_0 orthogonal to the vector field (in local coordinates generated by the exponential mapping). To do this, we discretize the problem and note that there exists a number $\sigma > 0$ such that a point $p \in M$ close to p_0 certainly belongs to $W^s(p_0)$ if and only if the distances of consecutive points of intersections of the positive semitrajectory of p with the sets $\exp_{p_k}(\mathcal{M}_k)$ to the points p_k do not exceed σ .

For suitably small $\mu > 0$, we find all sequences t_k and $z_k \in V_k$, the subspace of $T_{p_k}M$ orthogonal to $X(p_k)$, such that for $k \geq 0$

$$|t_k - 1| \leq \mu, \quad |z_k| \leq \mu, \quad y_{k+1} = \phi(t_k, y_k),$$

where $y_k = \exp_{p_k}(z_k)$. Thus we have to solve the equation

$$\exp_{p_{k+1}}(z_{k+1}) = \phi(t_k, \exp_{p_k}(z_k)), \quad k \geq 0$$

for t_k and $z_k \in V_k$ such that $|t_k - 1| \leq \mu$ and $|z_k| \leq \mu$.

We set it up as a problem in Banach spaces. By lemmas 4 and 6 the difference equation

$$z_{k+1} = B_k z_k, \quad z_k \in V_k$$

(recall that $B_k = P_{k+1}A_k$ and P_k is the orthogonal projection on \mathcal{M}_k with range V_k), has an exponential dichotomy on \mathbb{Z}_+ with projection (say) $Q_k : V_k \mapsto V_k$. Denote by $\mathcal{R}(Q_0)$ the range of Q_0 and note that $\mathcal{R}(Q_0) = \mathcal{E}^s$. Fix a positive number μ_0 and denote by V the space of sequences

$$\{z_k \in V_k, |z_k| \leq \mu_0, k \in \mathbb{Z}_+\}$$

and by $\ell^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\})$ the space of sequences $\{\zeta_k \in \mathcal{M}_{k+1}, k \in \mathbb{Z}_+\}$ with the usual norm.

Then the C^1 function

$$G : [1 - \mu_0, 1 + \mu_0]^{\mathbb{Z}_+} \times V \times \mathcal{R}(Q_0) \mapsto \ell^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\}) \times \mathcal{R}(Q_0)$$

given by

$$G(t, z, \eta) = (\{z_{k+1} - \exp_{p_{k+1}}^{-1}(\phi(t_k, \exp_{p_k}(z_k)))\}_{k \geq 0}, Q_0 z_0 - \eta)$$

is defined if μ_0 is small enough.

We want to solve the equation

$$G(t, z, \eta) = 0$$

for (t, z) as a function of η . It is clear that

$$G(1, 0, 0) = 0,$$

where the first argument of G is $\{1, 1, \dots\}$, the second argument is $\{0, 0, \dots\}$ and the right-hand side is $(\{0, 0, \dots\}, 0)$.

To apply the implicit function theorem, we must verify that

$$T = \frac{\partial G}{\partial(t, z)}(1, 0, 0)$$

is invertible. Note that if $(s, w) \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}) \times V$, then

$$T(s, w) = (\{w_{k+1} - X(p_{k+1})s_k - A_k w_k\}_{k \geq 0}, Q_0 w_0).$$

To show that T is invertible, we must show that

$$T(s, w) = (g, \eta)$$

has a unique solution (s, w) for all $(g, \eta) \in l^\infty(\mathbb{Z}_+, \{\mathcal{M}_{k+1}\}) \times \mathcal{R}(Q_0)$. So we need to solve the equations

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k + g_k, \quad k \geq 0$$

subject to

$$Q_0 w_0 = \eta.$$

If we multiply the difference equation by $X(p_{k+1})^*$ and solve for s_k , we obtain

$$s_k = -\frac{X(p_{k+1})^*}{|X(p_{k+1})|^2} [A_k w_k + g_k], \quad k \geq 0$$

and if we multiply it by P_{k+1} , we obtain

$$w_{k+1} = P_{k+1} A_k w_k + P_{k+1} g_k = B_k w_k + P_{k+1} g_k, \quad k \geq 0.$$

Now we know this last equation has a unique bounded solution $w_k \in V_k$, $k \geq 0$, satisfying $Q_0 w_0 = \eta$. Then the invertibility of T follows.

Thus we can apply the implicit function theorem to show that there exists $\mu > 0$ such that provided $|\eta|$ is sufficiently small, the equation $G(t, z, \eta) = 0$ has a unique solution $(t(\eta), z(\eta))$ such that $\|t - 1\|_\infty \leq \mu$, $\|z\|_\infty \leq \mu$. Moreover, $t(0) = 1$, $z(0) = 0$ and the functions $t(\eta)$ and $z(\eta)$ are C^1 .

The points $\exp_{p_0}(z_0(\eta))$ with small $|\eta|$ form a submanifold containing p_0 and contained in $W^s(p_0)$. Thus, the range of the derivative $z'_0(0)$ is contained in E^s .

Take an arbitrary vector $\xi \in \mathcal{E}^s$ and consider $\eta = \tau\xi, \xi \in \mathbb{R}$. Differentiating the equalities

$$z_{k+1}(\tau\xi) = \exp_{p_{k+1}}^{-1}(\phi(t_k(\tau\xi), \exp_{p_k}(z_k(\tau\xi)))), \quad k \geq 0,$$

and

$$Q_0 z_0(\tau\xi) = \tau\xi$$

with respect to τ at $\tau = 0$, we see that

$$s_k = \frac{\partial t_k}{\partial \eta}|_{\eta=0} \xi, \quad w_k = \frac{\partial z_k}{\partial \eta}|_{\eta=0} \xi \in V_k$$

are bounded sequences satisfying

$$w_{k+1} = A_k w_k + X(p_{k+1})s_k, \quad Q_0 w_0 = \xi.$$

Multiplying by P_{k+1} , we conclude that

$$w_{k+1} = B_k w_k, k \geq 0, \quad Q_0 w_0 = \xi.$$

It follows that $w_0 \in \mathcal{E}^s = \mathcal{R}(Q_0)$. Then $w_0 = Q_0 w_0 = \xi$. We have shown that the range of $z'_0(0)$ is exactly \mathcal{E}^s , and thus $\mathcal{E}^s \subset E^s$.

4 Lipschitz periodic shadowing

It is known that a vector field X is Ω -stable if and only if X satisfies Axiom A' and the no-cycle condition (see [29] and [30]). Thus, to prove Theorem 2, we prove the following two lemmas.

Lemma 7. *If a vector field X has the Lipschitz periodic shadowing property, then X satisfies Axiom A' and the no-cycle condition.*

Lemma 8. *If X satisfies Axiom A' and the no-cycle condition, then X has the Lipschitz periodic shadowing property.*

Lemma 7 is proved in Secs. 4.1-4.5; Lemma 8 is proved in Sec. 4.6.

The proof of Lemma 7 is divided into several steps.

We assume that X has the Lipschitz periodic shadowing property and establish the following statements.

1. Closed trajectories are uniformly hyperbolic.
2. Rest points are hyperbolic.
3. The chain-recurrent set coincides with the closure of the set of rest points and closed trajectories; rest points are separated from the remaining part of the chain-recurrent set.
4. The hyperbolic structure on the set of closed trajectories can be extended to the chain-recurrent set.
5. The no-cycle condition holds.

4.1 Uniform hyperbolicity of closed trajectories

Without loss of generality we can assume that $\mathcal{L} > 1$.

Let $x(t)$ be a nontrivial closed trajectory of period ω . Choose $n_1, n \in \mathbb{N}$ such that $\tau = n_1\omega/n \in [1/2, 1]$. Let $x_k = x(k\tau)$, $f(x) = \phi(\tau, x)$, and $A_k = Df(x_k)$. Note that $A_{k+n} = A_k$. Below we prove a statement similar to Lemma 2.

Lemma 9. *If $X \in \text{LipPerSh}$, then for any $b > 0$ there exists a constant K (the same for all closed trajectories $x(t)$ of X) such that for any sequence $b_k \in T_{x_k}M$ with $|b_k| < b$ there exist sequences $s_k \in \mathbb{R}$ and $v_k \in T_{x_k}M$ with the following properties:*

$$v_{k+1} = A_k v_k + X(x_{k+1})s_{k+1} + b_{k+1} \quad (26)$$

and

$$|s_k|, |v_k| \leq K. \quad (27)$$

Before we go to the proof of Lemma 9, we need to generalize the notion of discrete Lipschitz shadowing property. Let $d, \tau > 0$; we say that a sequence y_k is a τ -discrete d -pseudotrajectory if $\text{dist}(y_{k+1}, \phi(\tau, y_k)) < d$.

Let $\varepsilon > 0$; we say that a sequence x_k ε -shadows y_k if there exists a sequence $t_k > 0$ such that

$$\text{dist}(x_k, y_k) < \varepsilon, \quad |t_k - \tau| < \varepsilon, \quad x_{k+1} = \phi(t_k, x_k).$$

The following lemma can be proved similarly to Lemma 1.

Lemma 10. *If $X \in \text{LipPerSh}$, then there exist constants $d_0, L > 0$ such that for any $\tau \in [1/2, 1]$ and $d > 0$ and any periodic τ -discrete d -pseudotrajectory y_k with $d \leq d_0$ there exists a sequence x_k (not necessarily periodic) that Ld -shadows y_k .*

In the proof of Lemma 9, we use the following technical statement which is well-known in control theory [31],[32].

Lemma 11. *Let $B : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a linear operator such that the absolute values of its eigenvalues equal 1. Then for any $\Delta_0 \in \mathbb{R}^m$ and $\delta > 0$ there exists a number $R \in \mathbb{N}$ and a sequence $\delta_k \in \mathbb{R}^m$, $k \in [1, R]$, such that $|\delta_k| < \delta$ and the sequence $\Delta_k \in \mathbb{R}^m$ defined by*

$$\Delta_{k+1} = B\Delta_k + \delta_{k+1}, \quad k \in [0, R-1], \quad (28)$$

satisfies $\Delta_R = 0$.

Proof of Lemma 9. Fix an arbitrary sequence b_k with $|b_k| < b$ and a number $l \in \mathbb{N}$.

First we will find a number $l_1 > l$ and sequences c_k and Δ_k defined for $k \in [-ln, l_1n]$ such that $|c_k| < b$ and

$$\begin{aligned} c_k &= b_k, \quad k \in [-ln, ln], \\ \Delta_{k+1} &= A_k \Delta_k + c_{k+1}, \quad k \in [-ln, l_1n - 1], \\ \Delta_{-ln} &= \Delta_{l_1n}. \end{aligned} \tag{29}$$

Consider the operator $A : T_{x_0} \rightarrow T_{x_0}$ defined by $A = A_{n-1} \cdots A_0$.

The tangent space T_{x_0} can be represented in the form

$$T_{x_0} = E_0^s \oplus E_0^c \oplus E_0^u \tag{30}$$

so that the subspace E_0^s corresponds to the eigenvalues λ_j of A such that $|\lambda_j| < 1$, the subspace E_0^c corresponds to the eigenvalues λ_j such that $|\lambda_j| = 1$, and the subspace E_0^u corresponds to the eigenvalues λ_j such that $|\lambda_j| > 1$.

For any index k consider the decomposition $T_{x_k} = E_k^s \oplus E_k^c \oplus E_k^u$ as the image of decomposition (30) under the mapping $A_{k-1} \cdots A_0$.

In the coordinates corresponding to these decompositions, the matrices A_k can be represented in the following form:

$$A_k = \text{diag}(A_k^s, A_k^c, A_k^u).$$

Set $A_\sigma = A_{n-1}^\sigma \cdots A_0^\sigma$ for $\sigma = s, c, u$. Consider the corresponding coordinate representations $b_k = (b_k^s, b_k^c, b_k^u)$, $c_k = (c_k^s, c_k^c, c_k^u)$, and $\Delta_k = (\Delta_k^s, \Delta_k^c, \Delta_k^u)$ (and note that the values $|b_k^s|, |b_k^c|, |b_k^u|$ are not necessarily less than b).

Equations (29) are equivalent to the system

$$\Delta_{k+1}^s = A_k^s \Delta_k^s + c_{k+1}^s, \tag{31}$$

$$\Delta_{k+1}^c = A_k^c \Delta_k^c + c_{k+1}^c,$$

$$\Delta_{k+1}^u = A_k^u \Delta_k^u + c_{k+1}^u. \tag{32}$$

Set $c_k = b_k$ for $k \in [-ln, ln - 1]$.

Consider the sequence satisfying (31) with initial data $\Delta_{-ln}^s = 0$ and denote Δ_{ln}^s by a^s ; Consider the sequence satisfying (32) with initial data $\Delta_{ln}^u = 0$ and denote Δ_{-ln}^u by a^u .

There exist numbers $l_s, l_u > 1$ such that

$$|A^{-l}a^u| < b, \quad l \geq l_u, \quad |A^l a^s| < b, \quad l \geq l_s.$$

Set $\Delta_{-ln} = (0, 0, a^u)$; then the definition of a^s and a^u implies that $\Delta_{ln} = (a^s, C_1, 0)$ for some $C_1 \in E_0^c$.

Set $c_k = 0$ for $k \in [ln + 1, (l + l_s)n]$; then $\Delta_{(l+l_s)n} = (A_s^{l_s} a^s, C_2, 0)$ for some $C_2 \in E_0^c$.

Set $c_k = 0$ for $k \in [(l + l_s)n + 1, (l + l_s + 1)n - 1]$ and $c_k = (-A_s^{l_s+1} a^s, 0, 0)$ for $k = (l + l_s + 1)n$. Then $\Delta_{(l+l_s+1)n} = (0, C_3, 0)$ for some $C_3 \in E_0^c$.

Applying Lemma 11 to $A^c : E_0^c \rightarrow E_0^c$, we find a number R and a sequence δ_k with $|\delta_k| \leq b$ such that if

$$x_{i+1} = A^c x_i + \delta_{i+1}, \quad x_0 = \Delta_{(l+l_s+1)n}^c,$$

then $x_R = 0$. Then if we set for $i = 0, \dots, R-1$, $c_k = 0$ for $(l + l_s + i + 1)n + 1 \leq k \leq (l + l_s + i + 2)n - 1$ and $c_{(l+l_s+i+2)n} = (0, \delta_{i+1}, 0)$, we see that

$$\Delta_{(l+l_s+i+2)n}^c = A^c \Delta_{(l+l_s+i+1)n}^c + \delta_{i+1}, \quad i = 0, \dots, R-1,$$

so that $\Delta_{(l+l_s+R+1)n}^c = 0$; of course, the other two components of $\Delta_{(l+l_s+R+1)n}$ remain zero.

Set $c_k = 0$ for $k \in [(l + l_s + R + 1)n + 1, (l + l_s + R + 2)n - 1]$ and $c_k = (0, 0, A_u^{-l_u} a^u)$ for $k = (l + l_s + R + 2)n$; then $\Delta_{(l+l_s+R+2)n} = (0, 0, A_u^{-l_u} a^u)$. Finally, we set $c_k = 0$ for $k \in [(l + l_s + R + 2)n + 1, (l + l_s + R + 2 + l_u)n]$ and see that $\Delta_{(l+l_s+2+R+l_u)n} = (0, 0, a^u) = \Delta_{-ln}$. Thus, we have constructed the sequences mentioned in the beginning of the proof.

Taking d small enough, considering the periodic τ -discrete pseudotrajectory $y_k = \exp_{x_k}(d\Delta_k)$, and repeating the reasoning similar to that in the proof of Lemma 2, we can prove that relations (26) and (27) hold with $K = L(2b + 1)$ for $k \in [-ln, ln - 1]$.

After that, we repeat the reasoning used in the last two paragraphs of the proof of Lemma 2 to complete the proof of Lemma 9. \square

As in Sec. 3.3, we define \mathcal{M}_k, V_k, P_k , and $B_k = P_{k+1}A_k : V_k \rightarrow V_{k+1}$. Note that $B_k^{-1} = P_k A_k^{-1}$. Since M is compact, there exists a constant $N > 0$ such that $\|D\phi(\tau, x)\| < N$ for any $\tau \in [-1, 1]$ and $x \in M$. Hence, $\|A_k\|, \|A_k^{-1}\| < N$, and

$$\|B_k\|, \|B_k^{-1}\| < N. \quad (33)$$

The same reasoning as in the proof of Lemma 6 establishes the following statement.

Lemma 12. *There exists a constant $K > 0$ (the same for all closed trajectories $x(t)$) such that for every sequence $b_k \in V_k$ with $|b_k| \leq 1$ there exists a solution $v_k \in V_k$ of the system*

$$v_{k+1} = B_k v_k + b_{k+1}$$

such that

$$\|v_k\| \leq K.$$

A remark on page 26 of [22], Lemma 12 and the inequalities (33) imply that there exist constants $C_1 > 0$ and $\lambda_1 \in (0, 1)$ (the same for all closed trajectories) and a representation $V_k = E^s(x_k) \oplus E^u(x_k)$ such that

$$B_k E^s(x_k) = E^s(x_{k+1}), \quad B_k E^u(x_k) = E^u(x_{k+1}),$$

$$|B_{l+k} \cdots B_k v^s| \leq C_1 \lambda_1^l |v^s|, \quad v^s \in E^s(x_k), \quad l > 0, k \in \mathbb{Z}$$

$$|B_{-l+k}^{-1} \cdots B_k^{-1} v^u| \leq C_1 \lambda_1^l |v^u|, \quad v^u \in E^u(x_k), \quad l > 0, k \in \mathbb{Z}.$$

Remark 2. In fact in [22] exponential dichotomy with uniform constants was proved only on \mathbb{Z}^+ . However we can extend the corresponding inequalities to the whole of \mathbb{Z} by the periodicity of B_k .

Since $\tau \in [1/2, 1]$ and $\|D\phi(\tau, x)\| \leq N$ the above conditions imply that there exist constants $C_2 > 0$ and $\lambda_2 \in (0, 1)$ such that if $x(t)$ is a closed trajectory, then

$$|P_{\phi(t, x_0)} D\phi(t, x(t_0)) v^s| \leq C_2 \lambda_2^t |v^s|, \quad v^s \in E^s(x(t_0)), \quad t > 0, \quad t_0 \in \mathbb{R}, \quad (34)$$

$$|P_{\phi(-t, x_0)} D\phi(-t, x(t_0)) v^u| \leq C_2 \lambda_2^t |v^u|, \quad v^u \in E^u(x(t_0)), \quad t > 0, \quad t_0 \in \mathbb{R}, \quad (35)$$

where $P_{y \in M}$ is the orthogonal projection of $T_y M$ with kernel $X(y)$, $E^{s,u}(x(t_0)) = P_{\phi(t_0, x)} D\phi(t_0, x) E^{s,u}(x_0)$.

Remark 3. In particular, the above inequalities imply that $x(t)$ is a hyperbolic closed trajectory.

4.2 Hyperbolicity of the rest points

Let x_0 be a rest point. As in subsection 3.1 (using Lemma 9), we conclude that $D\phi(1, x_0)$ is hyperbolic; hence, x_0 is a hyperbolic rest point.

4.3 The rest points are separated from the remaining part of the chain-recurrent set

Denote by $\text{Per}(X)$ the set of rest points and points belonging to closed trajectories of a vector field X ; let $\mathcal{CR}(X)$ be the set of its chain-recurrent points. For a set $A \subset M$ denote by $\text{Cl } A$ the closure of A and by $B(a, A)$ its a -neighborhood.

Lemma 13. *If $X \in \text{LipPerSh}$, then $\text{Cl } \text{Per}(X) = \mathcal{CR}(X)$.*

Proof. If $y_0 \in \mathcal{CR}(X)$, then for any $d > 0$ there exists a periodic d -pseudotrajectory $g(t)$ such that $g(0) = y_0$.

Since $X \in \text{LipPerSh}$, there exists a point $x_d \in \text{Per}(X)$ such that $\text{dist}(x_d, y_0) < \mathcal{L}d$. Hence, $B(\mathcal{L}d, y_0) \cap \text{Per}(X) \neq \emptyset$ for arbitrary $d > 0$, which proves our lemma. \square

Lemma 14. *Let $X \in \text{LipPerSh}$ and let p be a rest point of X . Then $p \notin \text{Cl}(\mathcal{CR}(X) \setminus p)$.*

Proof. It has already been proved that all rest points of a vector field $X \in \text{LipPerSh}$ are hyperbolic; hence the set of rest points is finite. Assume that $p \in \text{Cl}(\mathcal{CR}(X) \setminus p)$. Then Lemma 13 implies that $p \in \text{Cl}(\text{Per}(X) \setminus p)$.

Denote by $W_{loc,a}^s(p)$ and $W_{loc,a}^u(p)$ the local stable and unstable manifolds of size a .

Since the rest point p is hyperbolic, there exists $\varepsilon \in (0, 1/2)$ such that if $x \in M$ and $\phi(t, x) \subset B(4\varepsilon, p)$, $t \geq 0$, then $x \in W_{loc,4\varepsilon}^s(p)$; if $\phi(t, x) \subset B(4\varepsilon, p)$, $t \leq 0$, then $x \in W_{loc,4\varepsilon}^u(p)$; and if $\phi(t, x) \subset B(4\varepsilon, p)$, $t \in \mathbb{R}$, then $x = p$.

Let $d_1 = \min(d_0, \varepsilon/\mathcal{L})$, where d_0 and \mathcal{L} are the constants from the definition of LipPerSh . Take a point $x_0 \in \text{Per}(X)$ (let the period of the trajectory of x_0 equal ω) and a number $T > 0$ and define the mapping

$$g_{x_0,T}(t) = \begin{cases} p, & t \in [-T, T], \\ \phi(t - T, x_0), & t \in (T, T + \omega), \end{cases}$$

for $t \in [-T, T + \omega]$. Continue this mapping periodically to the line \mathbb{R} .

There exists $d_2 < d_1$ depending only on d_1 and ν (see (3)) such that if $x_0 \in B(d_2, p)$, then $g_{x_0,T}(t)$ is a d_1 -pseudotrajectory for any $T > 0$. We fix such a point $x_0 \in B(d_2, p)$ and consider below pseudotrajectories $g_{x_0,T}$ with this fixed x_0 and with increasing numbers T .

By our assumptions, the pseudotrajectory $g_{x_0, T}$ can be ε -shadowed by the trajectory of a point $z_T \in \text{Per}(X)$ with reparametrization $\alpha_T(t)$:

$$\text{dist}(g_{x_0, T}(t), \phi(\alpha_T(t), z_T)) < \varepsilon. \quad (36)$$

Our choice of ε implies that there exist times $t_1, t_2 > 0$ such that

$$\text{dist}(p, \phi(t_1, x_0)) \in [2\varepsilon, 3\varepsilon], \quad \phi(t, x_0) \in B(4\varepsilon, p), \quad t \in [0, t_1],$$

$$\text{dist}(p, \phi(-t_2, x_0)) \in [2\varepsilon, 3\varepsilon], \quad \phi(t, x_0) \in B(4\varepsilon, p), \quad t \in [-t_2, 0].$$

We emphasize that the numbers t_1, t_2 depend on our choice of the point x_0 but not on our choice of T . Let

$$r_T = \phi(\alpha_T(T + t_1), z_T), \quad q_T = \phi(\alpha_T(-T - t_2), z_T).$$

Inequalities (36) and the following two relations imply that

$$\phi(t, q_T) \in B(5\varepsilon, p), \quad t \in [0, -\alpha_T(-T - t_2)], \quad (37)$$

$$\phi(t, r_T) \in B(5\varepsilon, p), \quad t \in [-\alpha_T(T + t_1), 0]. \quad (38)$$

Since $\mathcal{L}d_2 \leq \varepsilon < 1/2$ and t_1, t_2 are fixed, inequality (1) implies that if T is large enough, then

$$-\alpha_T(-T - t_2) \geq T/2, \quad \alpha_T(T + t_1) \geq T/2. \quad (39)$$

Since (37)-(39) imply that $\text{dist}(\phi(t, q_T), p) \leq 4\varepsilon$ for $0 \leq t \leq T/2$ and $\text{dist}(\phi(t, r_T), p) \leq 4\varepsilon$ for $0 \geq t \geq -T/2$ it follows that

$$\text{dist}(q_T, W_{loc, 4\varepsilon}^s(p)), \text{dist}(r_T, W_{loc, 4\varepsilon}^u(p)) \rightarrow 0, \quad T \rightarrow +\infty.$$

Since $q_T, r_T \in B(4\varepsilon, p) \setminus B(\varepsilon, p)$, we can choose sequences $q_n = q_{T_n} \rightarrow q$ and $r_n = r_{T_n} \rightarrow r$ such that $q, r \neq p$, $q \in W_{loc, 4\varepsilon}^s(p)$, and $r \in W_{loc, 4\varepsilon}^u(p)$.

Denote by $O(q_n)$ the (closed) trajectory of the point q_n .

From Remark 3 we know that $O(q_n)$ is a hyperbolic closed trajectory.

Passing to a subsequence, if necessary, we may assume that the values $\dim W^s(O(q_n))$ are the same for all n . Since

$$\dim W^s(O(q_n)) + \dim W^u(O(q_n)) = \dim M + 1$$

and

$$\dim W^s(p) + \dim W^u(p) = \dim M,$$

we see that at least one of the following inequalities holds:

$$\dim W^s(O(q_n)) > \dim W^s(p)$$

or

$$\dim W^u(O(q_n)) > \dim W^u(p).$$

Without loss of generality, we can assume that the first inequality holds (in the other case we note that $O(q_n) = O(r_n)$ and consider the vector field $-X$).

Denote $\sigma = \dim W^s(p)$. Consider the space $E_n^s = E^s(q_n)$ corresponding to inequalities (34), (35). Then the following holds

$$\dim E_n^s = \dim W^s(O(q_n)) - 1 \geq \sigma.$$

Passing to a subsequence, if necessary, we may assume that $E_n^s \rightarrow F^s \subset V_q$, where V_q is the subspace in $T_q M$ orthogonal to $X(q)$ (here and below, we consider convergence of linear spaces in the Grassman topology). Passing to the limit in inequalities (34), we conclude that

$$|P_{\phi(t,q)} D\phi(t,q) v^s| \leq C_2 \lambda_2^t |v^s|, \quad v^s \in F^s, \quad t > 0.$$

This inequality implies the inclusion $F^s \subset TW_q^s(p)$. Hence,

$$F^s \oplus \langle X(q) \rangle \subset T_q W^s(q),$$

and $\dim W^s(q) \geq \sigma + 1$. We get a contradiction which proves Lemma 12. \square

4.4 Hyperbolicity of the chain-recurrent set

Consider a point $y \in \mathcal{CR}(X)$ that is not a rest point. Lemma 13 implies that there exists a sequence $x_n \in \text{Per}(X)$ such that $x_n \rightarrow y$.

Consider the decomposition $V_{x_n} = E^s(x_n) + E^u(x_n)$ corresponding to inequalities (34), (35). Denote $E_n^{s,u} = E^{s,u}(x_n)$. Passing if necessary to a subsequence, we may assume that the dimensions $\dim E_n^s$ and $\dim E_n^u$ are the same for all n . Since y is not a rest point, $V_{x_n} \rightarrow V_y$.

Since inequalities (34) and (35) hold for all closed trajectories with the same constants C_2 and λ_2 , standard reasoning implies that the “angles” between E_n^s and E_n^u are uniformly separated from 0 (see, for instance, [3]). So passing if necessary to a subsequence, we may assume that $E_n^s \rightarrow E^s$ and $E_n^u \rightarrow E^u$.

Hence, $E^s \cap E^u = \{0\}$, $\dim(E^s + E^u) = \dim E^s + \dim E^u = \dim V_y$, and $E^s + E^u = V_y$. Estimates (34) and (35) for the points x_n imply similar estimates for y . Hence, the skew product flow (21) is hyperbolic, and Theorem 3 in Sacker and Sell [28] implies that $\mathcal{CR}(X)$ is hyperbolic.

4.5 No-cycle condition

In the previous two subsections we have proved that the vector field X (and its flow ϕ) satisfies Axiom A'. It is known that in this case, the nonwandering set of X can be represented as a disjoint union of a finite number of compact invariant sets (called basic sets):

$$\Omega(X) = \Omega_1 \cup \dots \cup \Omega_m, \quad (40)$$

where each of the sets Ω_i is either a hyperbolic rest point of X or a hyperbolic set on which X does not vanish and which contains a dense positive semi-trajectory.

The basic sets Ω_i have stable and unstable “manifolds”:

$$W^s(\Omega_i) = \{x \in M : \text{dist}(\phi(t, x), \Omega_i) \rightarrow 0, \quad t \rightarrow \infty\}$$

and

$$W^u(\Omega_i) = \{x \in M : \text{dist}(\phi(t, x), \Omega_i) \rightarrow 0, \quad t \rightarrow -\infty\}.$$

If Ω_i and Ω_j are basic sets, we write $\Omega_i \rightarrow \Omega_j$ if the intersection

$$W^u(\Omega_i) \cap W^s(\Omega_j)$$

contains a wandering point.

We say that X has a 1-cycle if there is a basic set Ω_i such that $\Omega_i \rightarrow \Omega_i$.

We say that X has a k -cycle if there are $k > 1$ basic sets

$$\Omega_{i_1}, \dots, \Omega_{i_k}$$

such that

$$\Omega_{i_1} \rightarrow \dots \rightarrow \Omega_{i_k} \rightarrow \Omega_{i_1}.$$

Lemma 15. *If $X \in \text{LipPerSh}$, then X has no cycles.*

Proof. To simplify the presentation, we prove that X has no 1-cycles (in the general case, the idea is essentially the same, but the notation is heavy).

To get a contradiction, assume that

$$p \in (W^u(\Omega_i) \cap W^s(\Omega_i)) \setminus \Omega(X).$$

Then there are sequences of times $j_m, k_m \rightarrow \infty$ as $m \rightarrow \infty$ such that

$$\phi(-j_m, p), \phi(k_m, p) \rightarrow \Omega_i, \quad m \rightarrow \infty.$$

Since the set Ω_i is compact, we may assume that

$$\phi(-j_m, p) \rightarrow q \in \Omega_i \quad \text{and} \quad \phi(k_m, p) \rightarrow r \in \Omega_i.$$

Since Ω_i contains a dense positive semi-trajectory, there exist points $s_m \rightarrow r$ and times $l_m > 0$ such that $\phi(l_m, s_m) \rightarrow q$ as $m \rightarrow \infty$.

Clearly, if we continue the mapping

$$g(t) = \begin{cases} \phi(t, p), & t \in [0, k_m], \\ \phi(t - k_m, s_m), & t \in [k_m, k_m + l_m], \\ \phi(t - j_m - k_m - l_m, p), & t \in [k_m + l_m, k_m + l_m + j_m], \end{cases}$$

periodically with period $k_m + l_m + j_m$, we get a periodic d_m -pseudotrajectory of X with $d_m \rightarrow 0$ as $m \rightarrow \infty$.

Since $X \in \text{LipPerSh}$, there exist points $p_m \in \text{Per}(X)$ (for m large enough) such that $p_m \rightarrow p$ as $m \rightarrow \infty$, and we get the desired contradiction with the assumption that $p \notin \Omega(X)$. The lemma is proved. \square

4.6 Ω -stability implies Lipschitz periodic shadowing

The proof of Lemma 8 is similar to the corresponding proof in [18], where the case of diffeomorphisms is considered. In the present article we give the most important steps and leave the details to the reader.

Proof of Lemma 8. Let us formulate several auxiliary definitions and statements.

Let us say that a vector field X has the Lipschitz shadowing property on a set U if there exist positive constants \mathcal{L}, d_0 such that if $g(t)$ with $\{g(t) : t \in \mathbb{R}\} \subset U$ is a d -pseudotrajectory (in our standard sense:

$$\text{dist}(g(\tau + t), \phi(t, g(\tau))) < d, \quad \tau \in \mathbb{R}, t \in [0, 1])$$

with $d \leq d_0$, then there exists a point $p \in U$ and a reparametrization α satisfying inequality (1) such that

$$\text{dist}(g(t), \phi(\alpha(t), p)) < \mathcal{L}d, \quad t \in \mathbb{R}. \quad (41)$$

We say that a vector field X is expansive on a set U if there exist positive numbers a (expansivity constant) and δ such that if two trajectories $\{\phi(t, p) :$

$t \in \mathbb{R}$ and $\{\phi(t, q) : t \in \mathbb{R}\}$ belong to U and there exists a continuous real-valued function $\alpha(t)$ such that

$$\text{dist}(\phi(\alpha(t), q), \phi(t, p)) \leq a, \quad t \in \mathbb{R},$$

then $p = \phi(\tau, q)$ for some real $\tau \in (-\delta, \delta)$.

Let X be an Ω -stable vector field. Consider the decomposition (40) of $\Omega(X)$. We will refer to the following well-known statement [1].

Theorem 4. *If Ω_i is a basic set, then there exists a neighborhood U of Ω_i such that X has the Lipschitz shadowing property on U and is expansive on U .*

We also need the following two lemmas. Analogs of these lemmas were proved for diffeomorphisms in [33]; the proofs for flows are the same.

Lemma 16. *For any neighborhood U of the nonwandering set $\Omega(X)$ there exist positive numbers B, d_1 such that if $g(t)$ is a d -pseudotrajectory of ϕ with $d \leq d_1$ and*

$$g(t) \notin U, \quad t \in [\tau, \tau + l],$$

for some $l > 0$ and $\tau \in \mathbb{R}$, then $l \leq B$.

Lemma 17. *Assume that the vector field X is Ω -stable. Let U_1, \dots, U_m be disjoint neighborhoods of the basic sets $\Omega_1, \dots, \Omega_m$. There exist neighborhoods $V_j \subset U_j$ of the sets Ω_j and a number $d_2 > 0$ such that if $g(t)$ is a d -pseudotrajectory of X with $d \leq d_2$, $g(\tau) \in V_j$ and $g(\tau + t_0) \notin U_j$ for some $j \in \{1, \dots, m\}$, some $\tau \in \mathbb{R}$ and some $t_0 > 0$, then $g(\tau + t) \notin V_j$ for $t \geq t_0$.*

Now we pass to the proof itself.

Apply Theorem 4 and Lemmas 16, 17 and find disjoint neighborhoods W_1, \dots, W_m of the basic sets $\Omega_1, \dots, \Omega_m$ such that

- (i) X has the Lipschitz shadowing property on each W_j with the same constants \mathcal{L}, d_0^* ;
- (ii) X is expansive on each W_j with the same expansivity constants a, δ .

Find neighborhoods V_j, U_j of Ω_j (and reduce d_0^* , if necessary) so that the following properties are fulfilled:

- $V_j \subset U_j \subset W_j$, $j = 1, \dots, m$;
- the statement of Lemma 17 holds for V_j and U_j with some $d_2 > 0$;

- the $\mathcal{L}d_0^*$ -neighborhoods of U_j belong to W_j .

Apply Lemma 16 to find the corresponding constants B, d_1 for the neighborhood $V_1 \cup \dots \cup V_m$ of $\Omega(X)$.

We claim that X has the Lipschitz periodic shadowing property with constants \mathcal{L}, d_0 , where

$$d_0 = \min \left(d_0^*, d_1, d_2, \frac{a}{2\mathcal{L}} \right).$$

Take a μ -periodic d -pseudotrajectory $g(t)$ of X with $d \leq d_0$. Without loss of generality we can assume that $\mu > \delta$ (since μ is not necessarily the minimal period). Lemma 16 implies that there exists a neighborhood V_j such that the pseudotrajectory $g(t)$ intersects V_j ; shifting time, we may assume that $g(0) \in V_j$.

In this case, $\{g(t) : t \in \mathbb{R}\} \subset U_j$. Indeed, if $g(t_0) \notin U_j$ for some t_0 , then $g(t_0 + k\mu) \notin U_j$ for all k . It follows from Lemma 17 that if $t_0 + k\mu > 0$, then $g(t) \notin V_j$ for $t \geq t_0 + k\mu$, and we get a contradiction with the periodicity of $g(t)$ and the inclusion $g(0) \in V_j$.

Thus, there exists a point p such that inequalities (41) hold for some reparametrization α satisfying inequality (1). Let us show that either p is a rest point or the trajectory of p is closed. By the choice of U_j and W_j , $\phi(t, p) \in W_j$ for all $t \in \mathbb{R}$. Let $q = \phi(\mu, p)$.

Inequalities (41) and the periodicity of $g(t)$ imply that

$$\text{dist}(g(t), \phi(\alpha(t + \mu) - \mu, q)) =$$

$$\text{dist}(g(t + \mu), \phi(\alpha(t + \mu), p)) \leq \mathcal{L}d, \quad t \in \mathbb{R}.$$

Thus,

$$\text{dist}(\phi(\alpha(t), p), \phi(\alpha(t + \mu) - \mu, q)) \leq 2\mathcal{L}d \leq a, \quad t \in \mathbb{R},$$

which implies that

$$\text{dist}(\phi(\theta, p), \phi(\beta(\theta), q)) \leq 2\mathcal{L}d \leq a, \quad \theta \in \mathbb{R},$$

where $\beta(\theta) = \alpha(\alpha^{-1}(\theta) + \mu) - \mu$.

Since $\phi(t, p) \in W_j$ for all $t \in \mathbb{R}$, our expansivity condition on W_j implies that $q = \phi(\tau, p)$ for some $\tau \in (-\delta, \delta)$.

This completes the proof. \square

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